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Construction of a stationary queue with impatient customers

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Abstract

In this paper, we study the stability of queues with impatient customers. Under general stationary ergodic assumptions, we first provide some conditions for such a queue to be regenerative (i.e. to empty a.s. an infinite number of times). In the particular case of a single server operating in First in, First out, we prove the existence (in some cases, on an enlarged probability space) of a stationary workload. This is done by studying a non-monotonic stochastic recursion under the Palm settings, and by stochastic comparison of stochastic recursions.

keywords: Stochastic recursions; Stationary solutions ; Queues with impatience ; Renovative events ; Enlargement of probability space.

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1 Introduction

In this paper, we address the question of stability for queueing systems with impatient customers: the customers agree to wait for their service only during a limited period of time. They are discarded from the system provided that their patience ends before they could reach the service booth. Such models are particularly adequate to describe operating systems under sharp delay requirements: multimedia and time sensitive telecommunication and computer networks, on-line audio/video traffic flows, call centres or supply chains.

We first give conditions of regenerativity, i.e., for the 0 state to be recurrent for the congestion process of the system. Then, we construct explicitly a stationary state for these systems in the particular case of a single server obeying the FIFO (First In, First Out) discipline. In that purpose, we study a stochastically recursive sequence representing the workload seen by an arriving customer. This sequence and its dynamics have been thoroughly studied in the GI/GI/1 case in [6] and [3]. In the G/G/1 context, the workload sequence is driven by a non-monotonic recursive equation (eq. (9)), and hence a construction of Loynes' type, using a backwards recurrence scheme, is not possible. We thus use more recent and sophisticated techniques to construct a stationary workload: (i) Borovkov's theory of renovating events (see [8], [9], [4]) provides a sufficient condition for the existence and uniqueness of a finite stationary workload. Under

this condition, we can thus construct a stationary loss probability π, and provide bounds for π (eq. (11)). (ii) We prove in whole generality the existence of a stationary workload on the enriched probability space $\Omega \times \mathbb{R}+$ (where Ω is the Palm probability space of reference) using Anantharam and Konstantopoulos' construction (see [1], [2]), which is based on tightness techniques.

In both cases, we use the fact that the workload sequence is strongly dominated by another one, driven by a monotonic recursive equation (eq. (1)). Then the coupling of the dominating sequence with a unique stationary state (which is proven by Loynes' scheme) allows us to construct the stationary state of the dominated sequence.

This paper is organized as follows. In section 2 we precise our basic assumptions, and solve eq. (1) in the stationary ergodic framework, a result that will be used in the sequel. We present the queue with impatient customers in section 3. In section 4, we provide conditions for the regenerativity of the system. In section 5, we construct a stationary workload in the FIFO case: we provide a sufficient condition for the existence and uniqueness in 5.1, and prove the existence of the stationary workload on an enriched probability space in 5.3.

2 Preliminaries

Consider a probability space $\left(\Omega, \mathcal{F}, \mathbf{P}^0\right)$, embedded with the measurable bijective flow θ (denote θ^{-1}, its measurable inverse). Suppose that $\mathbf{P}^0$ is stationary and ergodic under θ, *i.e.* for all $\mathfrak{A} \in \mathcal{F}$, $\mathbf{P}^0\left[\theta^{-1}\mathfrak{A}\right] = \mathbf{P}^0\left[\mathfrak{A}\right]$ and all $\mathfrak{A}$ that is θ-invariant (i.e. such that $\theta\mathfrak{A} = \mathfrak{A}$) is of probability 0 or 1. Note that according to these axioms, all θ-contracting event (such that $\mathbf{P}^0\left[\mathfrak{A}^c \cap \theta^{-1}A\right] = 0$) is θ-invariant. We denote for all $n \in \mathbb{N}$, $\theta^n = \theta \circ \theta \circ ... \circ \theta$ and $\theta^{-n} = \theta^{-1} \circ \theta^{-1} \circ ... \circ \theta^{-1}$, and say that two sequences of r.v. $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ couple when there exists a $\mathbf{P}^0$-a.s. finite rank N such that they coincide for all $n \geq N$. We say that there is strong backwards coupling between $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y \circ \theta^n\}_{n\in\mathbb{N}}$ provided that for some $\mathbf{P}^0$-a.s. finite τ, $X_n \circ \theta^{-n} = Y$ for any $n \geq \tau$.

Let α and β be two integrable $\mathbb{R}+$-valued r.v., and denote for all $n \in \mathbb{Z}$, $\alpha_n = \alpha \circ \theta^n$ and $\beta_n = \beta \circ \theta^n$. Let Z be an a.s. finite $\mathbb{R}+$-valued r.v., and consider the following stochastically recursive sequence.

$$\begin{cases} Y_0^Z &= Z, \\ Y_{n+1}^Z &= \left[\max\left\{Y_n^Z, \alpha_n\right\} - \beta_n\right]^+ \text{ for all } n \in \mathbb{N}. \end{cases}$$

Then, $\left\{Y_n^Y\right\}_{n\in\mathbb{N}}$ is a stationary version of this sequence provided that the r.v. Y is a solution to

$$Y \circ \theta = \left[\max\left\{Y, \alpha\right\} - \beta\right]^+. \tag{1}$$

We have the following result.

Lemma 2.1
There exists a unique $\mathbf{P}^0$-a.s. finite solution Y of (1), given by

$$Y := \left[\sup_{j\in\mathbb{N}^*}\left(\alpha_{-j} - \sum_{i=1}^{j}\beta_{-i}\right)\right]^+. \tag{2}$$

Moreover, for all $\mathbf{P}^0$-a.s. finite and nonnegative r.v. Z, the sequence $\left\{Y_n^Z\right\}_{n\in\mathbb{N}}$ couples with $\{Y\circ\theta^n\}_{n\in\mathbb{N}}$, and there exists $\mathbf{P}^0$-a.s. an infinity of indices such that $Y_n^Z=0$ if and only if

$$\mathbf{P}^0\left[Y=0\right]=0. \tag{3}$$

Proof. Equation (1) can be handled by Loynes's construction (see [11], [5]) since the mapping $[\max\{.,\alpha\}-\beta]^+\circ\theta^{-1}$ is $\mathbf{P}^0$-a.s. continuous and nondecreasing. Hence Y classically reads as the $\mathbf{P}^0$-a.s. limit of Loynes's sequence, defined by $\left\{Y_n^0\circ\theta^{-n}\right\}_{n\in\mathbb{N}}$. It is routine to check from Birkhoff's ergodic theorem that Y is $\mathbf{P}^0$-a.s. finite.

The coupling property follows from the fact that for all non-negative r.v. Z that is $\mathbf{P}^0$-a.s finite (and in particular, for $Y=Z$),

$$\left\{Y_n^Z\neq Y_n^0 \text{ for all } n\in\mathbb{N}\right\}=\left\{Y_n^Z=Z-\sum_{i=0}^{n-1}\beta_i>0 \text{ for all } n\in\mathbb{N}\right\},$$

which is of probability 0 from Birkhoff's theorem. The last statement is a classical consequence of this coupling property under ergodic assumptions. $\square$

3 The model

Let us consider a queue with impatient customers until the beginning of service G/G/s/s+G (according to Barrer's notation, see [7]): on the probability space $(\Omega,\mathcal{F},\mathbf{P})$, furnished with the measurable bijective flow $(\theta_t)_{t\in\mathbb{R}}$ under which $\mathbf{P}$ is stationary and ergodic, consider the θ_t-compatible point process N, whose points $\{T_n\}_{n\in\mathbb{Z}}$ represent the arrivals of the customers $\{C_n\}_{n\in\mathbb{Z}}$, with the convention that T_0 is the last arrival before time $t=0$. The interarrivals, denoted for all $n\in\mathbb{Z}$ by $\xi_n=T_{n+1}-T_n$, hence form a stationary sequence of integrable r.v., and we denote $\xi:=\xi_0=T_1-T_0$. The process N is marked by the stationary sequence $\{\sigma_n\}_{n\in\mathbb{Z}}$ denoting the service durations requested by the customers, and the integrable r.v. $\sigma:=\sigma_0$ is the service duration of customer C_0. The customers enter a system with s non-idling servers and of infinite capacity, and are impatient until the beginning of their service, in that they leave the system if they do not reach the service booth before a given deadline. In other words, customer C_n agrees to wait in line for a given period of time, say D_n (his initial *patience*) and if there is no server available during this period, he leaves the system forever at time T_n+D_n. $\{D_n\}_{n\in\mathbb{Z}}$ is a sequence of integrable and non-negative marks of $(N_t)_{t\geq 0}$, and $D:=D_0$ is the initial time credit of C_0. We denote for all $t\in\mathbb{R}$, $\mathcal{X}_t$ the number of customers in the system (or *congestion*) at t. The servers follow a non-preemptive service discipline. We therefore consider that a customer can not be eliminated anymore as soon as he enters the service booth, even if his deadline is reached during his service.

Let us denote $\left(\Omega,\mathcal{F},\mathbf{P}^0,\theta\right)$, the Palm probability space of $N(\sigma,D)$, where $\theta:=\theta_{T_1}$ is the associated bijective flow. In particular, $\mathbf{P}^0$ is stationary and ergodic under θ, and for all $n\in\mathbb{Z}$,

$$\xi_n=\xi\circ\theta^n\,,\ \sigma_n=\sigma\circ\theta^n \text{ and } D_n=D\circ\theta^n.$$

4 Regenerativity

According to the assumptions made above, the total sojourn time of customer C_n does not exceed $D_n + \sigma_n$, i.e. the sum of his initial patience and the time necessary for his service. On the other hand, it is at least equal to $\min\{\sigma_n, D_n\}$, i.e. the time needed for him to be lost, or immediately served. Hence, provided that C_n entered the system before t ($T_n \le t$) and even though he already left the system before t, his *remaining maximal sojourn time* at t (i.e. the remaining time before his latest possible departure time, if not already reached) is given by $[\sigma_n + D_n - (t - T_n)]^+$, whereas his *remaining minimal sojourn time* at t (i.e. the remaining time before his earliest possible departure time, if not already reached) is given by $[\min\{\sigma_n, D_n\} - (t - T_n)]^+$. Hence the *largest remaining maximal sojourn time* (LRMST for short) at t among all the customers entered before t is given by

$$\mathcal{L}_t := \max_{n=1}^{N_t} [\sigma_n + D_n - (t - T_n)]^+$$

and the largest remaining minimal sojourn time (LRmST for short) at t, by

$$\mathcal{M}_t := \max_{n=1}^{N_t} [\min\{\sigma_n, D_n\} - (t - T_n)]^+ .$$

The two processes $(\mathcal{L}_t)_{t\ge 0}$ and $(\mathcal{M}_t)_{t\ge 0}$ evolve according to the following dynamics: they decrease at unit rate between arrival times, and equal the initial maximal (resp. minimal) sojourn time of C_n at his arrival time T_n, provided that it is larger than the value of the process just before T_n. In other words, for all $n \in \mathbb{Z}$ and all $t \in [T_n, T_{n+1})$

$$\mathcal{L}_t = [\max\{\mathcal{L}_{T_n-}, \sigma_n + D_n\} - (t - T_n)]^+ ,$$

$$\mathcal{M}_t = [\max\{\mathcal{M}_{T_n-}, \min\{\sigma_n, D_n\}\} - (t - T_n)]^+ .$$

Define for all finite nonnegative r.v. Y and Z and all $n \in \mathbb{N}$, $L_n^Y := \mathcal{L}_{T_n-}$ and $M_n^Z := \mathcal{M}_{T_n-}$, the LRMST (resp. LRmST) just before the arrival of customer C_n, provided that $\mathcal{L}_{T_0-} = Y$ (resp. $\mathcal{M}_{T_0-} = Z$). The processes $(\mathcal{L}_t)_{t\in\mathbb{R}}$ and $(\mathcal{M}_t)_{t\in\mathbb{R}}$ have rcll paths, hence we have the following recursive equations.

$$L_{n+1}^Z = \left[\max\left\{L_n^Z, \sigma_n + D_n\right\} - \xi_n\right]^+ , \tag{4}$$

$$M_{n+1}^Z = \left[\max\left\{M_n^Z, \min\{\sigma_n, D_n\}\right\} - \xi_n\right]^+ .$$

On $\left(\Omega, \mathbf{P}^0\right)$ the two latter equations are of type (1), hence Lemma 2.1 implies that for any Y and Z, $\left\{L_n^Y\right\}_{n\in\mathbb{N}}$ and $\left\{M_n^Z\right\}_{n\in\mathbb{N}}$ couple respectively with $\{L \circ \theta^n\}_{n\in\mathbb{N}}$ and $\{M \circ \theta^n\}_{n\in\mathbb{N}}$, where L and M are given by

$$L = \left[\sup_{j\in\mathbb{N}^*} \left(\sigma_{-j} + D_{-j} - \sum_{i=1}^{j} \xi_{-i}\right)\right]^+ , \tag{5}$$

$$M = \left[\sup_{j\in\mathbb{N}^*} \left(\min\{\sigma_{-j}, D_{-j}\} - \sum_{i=1}^{j} \xi_{-i}\right)\right]^+ . \tag{6}$$

In particular, for any initial conditions Y and Z, there are $\mathbf{P}$-a.s. an infinity of indices such that $L_n^Y = 0$ if and only if $\mathbf{P}^0[L=0] > 0$, and an infinity of indices such that $M_n^Z = 0$ if and only if $\mathbf{P}^0[M=0] > 0$. Remarking now that for all initial conditions and all $t \geq 0$,

$$\{\mathcal{L}_t = 0\} \subseteq \{\mathcal{X}_t = 0\} \subseteq \{\mathcal{M}_t = 0\},$$

we obtain

Theorem 4.1
The G/G/s/s+G queue is regenerative (i.e. it empties $\mathbf{P}^0$-a.s. an infinite number of times) if

$$\mathbf{P}^0\left[\sup_{j\in\mathbb{N}^*}\left(\sigma_{-j} + D_{-j} - \sum_{i=1}^{j}\xi_{-i}\right) \leq 0\right] > 0, \tag{7}$$

and only if

$$\mathbf{P}^0\left[\sup_{j\in\mathbb{N}^*}\left(\min\{\sigma_{-j}, D_{-j}\} - \sum_{i=1}^{j}\xi_{-i}\right) \leq 0\right] > 0.$$

5 The FIFO case

Let us now consider the special case, where there is one server obeying the FIFO (*First in, first out*) discipline. Denote for all $t \in \mathbb{R}$, $\mathcal{W}_t$ the workload submitted to the server at time t, i.e. the quantity of work he still has to achieve at this time, in time unit. The process $(\mathcal{W}_t)_{t\in\mathbb{R}}$ has rcll paths, and we define for all n, $W_n = \mathcal{W}_{T_n^-}$. Its value at t equals the work brought by the customers arrived up to t, and who will eventually be served, since the other ones won't ever reach the server. Under the FIFO discipline, the served customers are those who find a workload less than their patience upon arrival. In-between arrival times, $(\mathcal{W}_t)_{t\in\mathbb{R}}$ decreases at unit rate. Hence, for all $n \in \mathbb{Z}$ and $t \in [T_n, T_{n+1})$

$$\mathcal{W}_t = \left[\mathcal{W}_{T_n-} + \sigma_n \mathbf{1}_{\{\mathcal{W}_{T_n-} \leq D_n\}} - (t - T_n)\right]^+,$$

whereas the workload sequence is driven by the recursive equation

$$W_{n+1} = \left[W_n + \sigma_n \mathbf{1}_{\{W_n \leq D_n\}} - \xi_n\right]^+. \tag{8}$$

For all $n \in \mathbb{N}$ and all finite non-negative r.v. Z, let W_n^Z be the workload seen by C_n upon arrival, provided that $W_0^Z = Z$. In addition to the previous result of regenerativity (Theorem 4.1), we investigate in this particular case the existence and uniqueness of a stationary version for the stochastic recursion (8), i.e., of a finite r.v. W such that $W_n^W = W \circ \theta^n$, $n \in \mathbb{N}$, which implies that

$$W \circ \theta = \left[W + \sigma \mathbf{1}_{\{W \leq D\}} - \xi\right]^+. \tag{9}$$

The recursive equation (9) is not monotonic in the state variable, in which case a construction of Loynes's type becomes fruitless. In the following sections, we propose two methods to circumvent this difficulty. Let us first remark that

Lemma 5.1
For all $n \in \mathbb{N}$, $W_n^0 \leq L_n^0$, $\mathbf{P}^0$-a.s..

Proof. For all $n \in \mathbb{N}$, on the event $\left\{W_n^0 \leq L_n^0\right\}$,

$$\begin{aligned} W_{n+1}^0 &= [W_n^0 - \xi_n]^+ \mathbf{1}_{W_n^0 > D_n} + [W_n^0 + \sigma_n - \xi_n]^+ \mathbf{1}_{W_n^0 \leq D_n} \\ &\leq [L_n^0 - \xi_n]^+ \mathbf{1}_{W_n^0 > D_n} + [D_n + \sigma_n - \xi_n]^+ \mathbf{1}_{W_n^0 \leq D_n} \\ &\leq [\max(L_n^0, \sigma_n + D_n) - \xi_n]^+ \mathbf{1}_{W_n^0 > D_n} + [\max(L_n^0, \sigma_n + D_n) - \xi_n]^+ \mathbf{1}_{W_n^0 \leq D_n} \\ &= L_{n+1}^0. \end{aligned} \tag{10}$$

Since $L_0^0 = W_0^0$, by induction $L_n^0 \geq W_n^0$ for all $n \in \mathbb{N}$, $\mathbf{P}^0$-a.s.. $\square$

5.1 Sufficient condition

In many examples, and even when monotonicity is not granted, a stochastic recurrence may admit a (possibly unique) stationary regime, provided that the sequence 'regenerates' in a constant manner. This can be proved using Borovkov's theory of renovating events.

Theorem 5.2
If (7) holds, then (9) admits a unique finite solution W, which is such that $M \leq W \leq L$, $\mathbf{P}^0$-a.s., where L and M are defined respectively by (5) and (6). Moreover, for any finite initial condition Z, there is strong backwards coupling for $\left\{W_n^Z\right\}_{n\in\mathbb{N}}$ with $\{W \circ \theta^n\}_{n\in\mathbb{N}}$.

Proof. Existence. From Lemma 5.1, for all $n \geq 0$, $W_n^0 \leq L_n^0 \leq L_n^L = L \circ \theta^n$, $\mathbf{P}^0$-a.s.. Hence denoting $\mathfrak{A}_n$, the event $\{L \circ \theta^n = 0\}$, $\{\mathfrak{A}_n\}_{n\in\mathbb{N}}$ is a sequence of renovating events of length 1 for the sequence $\left\{W_n^0\right\}_{n\in\mathbb{N}}$ since $\mathfrak{A}_n \subseteq \left\{W_n^0 = 0\right\}$ for any n (see [5], p.115, [8], [9]). Moreover, this sequence is stationary in the sense that for all $n \geq 0$, $\mathfrak{A}_n = \theta^{-n}\mathfrak{A}_0$, where $\mathfrak{A}_0 = \{L = 0\}$. Hence since (7) amounts to $\mathbf{P}^0[\mathfrak{A}_0] > 0$, this is from [5], Theorem 2.5.3., a sufficient condition for the existence of a solution to (9).

Uniqueness. Let W be a solution of (9). Then, we have $\mathbf{P}^0[W \leq D] > 0$. Indeed, if $W > D$, $\mathbf{P}^0$-a.s. (which impies in particular that $W \circ \theta > 0$, $\mathbf{P}^0$-a.s.), then $W \circ \theta = W - \xi$, $\mathbf{P}^0$-a.s., which is absurd since $\mathbf{E}^0[W - W \circ \theta] = 0$ from the ergodic Lemma. On another hand, in view of the inequalities (10), $\{W \leq L\}$ is clearly θ-contracting, whereas

$$0 < \mathbf{P}^0[W \leq D] \leq \mathbf{P}^0[W \circ \theta \leq L \circ \theta] = \mathbf{P}^0[W \leq L].$$

Hence $\{W \leq L\}$ is $\mathbf{P}^0$-almost sure, thus $\left\{W_n^W\right\}_{n\in\mathbb{N}} = \{W \circ \theta^n\}_{n\in\mathbb{N}}$ admits $\{\mathfrak{A}_n\}_{n\in\mathbb{N}}$ as a stationary sequence of renovating events of length 1. From [5], Cor. 2.5.1, $\mathbf{P}^0[\mathfrak{A}_0] > 0$ implies the coupling property.

Finally, to prove that $M \leq W$, $\mathbf{P}^0$-a.s., remark that on $\{M \leq W\}$,

$$
\begin{aligned}
M \circ \theta &= [M-\xi]^+ \mathbf{1}_{M \geq \min\{\sigma,D\}} + [\min\{\sigma,D\}-\xi]^+ \mathbf{1}_{M<\min\{\sigma,D\}} \\
&\leq [W-\xi]^+ \mathbf{1}_{M \geq \min\{\sigma,D\}} + [\sigma-\xi]^+ \mathbf{1}_{W \leq D} \mathbf{1}_{M<\min\{\sigma,D\}} \\
&\quad + [D-\xi]^+ \mathbf{1}_{W>D} \mathbf{1}_{M<\min\{\sigma,D\}} \\
&\leq [W+\sigma \mathbf{1}_{W \leq D}-\xi]^+ \mathbf{1}_{M \geq \min\{\sigma,D\}} + [W+\sigma-\xi]^+ \mathbf{1}_{W \leq D} \mathbf{1}_{M<\min\{\sigma,D\}} \\
&\quad + [W-\xi]^+ \mathbf{1}_{W>D} \mathbf{1}_{M<\min\{\sigma,D\}} \\
&= W \circ \theta \mathbf{1}_{M \geq \min\{\sigma,D\}} + W \circ \theta \mathbf{1}_{W \leq D} \mathbf{1}_{M<\min\{\sigma,D\}} + W \circ \theta \mathbf{1}_{W>D} \mathbf{1}_{M<\min\{\sigma,D\}} \\
&= W \circ \theta.
\end{aligned}
$$

Hence, $\{M \leq W\}$ is θ-contracting. On another hand, we have

$$\mathbf{P}^0 \left[M \leq \min\{\sigma,D\}\right] > 0,$$

since $M > \min\{\sigma,D\}$, $\mathbf{P}^0$-a.s. would imply in particular that $M \circ \theta > 0$, $\mathbf{P}^0$-a.s., and hence on a $\mathbf{P}^0$-almost sure event $M \circ \theta = M - \xi$, which is absurd, again in view of the Ergodic Lemma. But on $\{M \leq \min\{\sigma,D\}\}$,

$$M \circ \theta = [\min\{\sigma,D\}-\xi]^+ \leq [\sigma-\xi]^+ \mathbf{1}_{W \leq D} + [D-\xi]^+ \mathbf{1}_{W>D} \leq W \circ \theta,$$

hence

$$0 < \mathbf{P}^0 \left[M \leq \min\{\sigma,D\}\right] \leq \mathbf{P}^0 \left[M \circ \theta \leq W \circ \theta\right] = \mathbf{P}^0 \left[M \leq W\right],$$

which concludes the proof. □

5.2 Some applications

It is a very classical argument, that due to the FIFO discipline, the construction of the stationary versions of some quantities of interest can be derived from that of the workload sequence. In particular, provided that (7) holds, one can construct a congestion process and a departure process (without distinction between departures due to reneging and service completions) that are jointly compatible with the arrival process $(N_t)_{t \in \mathbb{R}}$. Let us remark, that under condition (7) there exists also a stationary *loss probability*, denoted π, which is the probability that the waiting time proposed to a customer exceeds his initial patience, at equilibrium. This reads

$$\pi = \mathbf{P}^0 \left[W > D\right].$$

With Theorem 5.2 in hands, we have in particular that

$$\mathbf{P}^0 \left[M > D\right] \leq \pi \leq \mathbf{P}^0 \left[L > D\right], \tag{11}$$

where L and M are given by (5) and (6), respectively.

5.3 Weak stationarity

In this section, condition (7) is no longer assumed to hold. We show how the techniques developed in [1] may allow us to construct a stationary workload

for the queue, on an enriched probability space. Again, this is done using the stochastic comparison with the LRMST sequence (Lemma 5.1). Let $\phi\{.\}$ be the measurable random map from $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$ into itself (we denote $\phi \in \mathfrak{M}(\mathbb{R})$) defined by

$$\phi(\omega)\{y\} = \left[y + \sigma(\omega)\mathbf{1}_{\{y \leq D(\omega)\}} - \xi(\omega)\right]^+ .$$

We work on the enlarged probability space $\Omega \times \mathbb{R}$, on which we define the shift

$$\tilde{\theta}(\omega, x) = (\theta\omega, \phi(\omega)\{x\}) .$$

We then have the following result.

Theorem 5.3

The stochastic recursion (9) admits a *weak solution*, that is, a $\tilde{\theta}$-invariant probability $\tilde{\mathbf{P}}^0$ on $\Omega \times \mathbb{R}$ whose Ω-marginal is $\mathbf{P}^0$. Therefore, on $(\Omega \times \mathbb{R})$ there exists a $\mathbb{R} \times \mathfrak{M}(\mathbb{R})$-valued r.v. $\left(\tilde{W}, \tilde{\phi}\right)$ satisfying

$$\tilde{W} \circ \tilde{\theta} = \tilde{\phi}\left\{\tilde{W}\right\} .$$

In particular, $\left\{\left(\tilde{W}, \tilde{\phi}\right) \circ \tilde{\theta}^n\right\}_{n \in \mathbb{N}}$ is stationary under $\tilde{\mathbf{P}}^0$, and $\left\{\tilde{\phi} \circ \tilde{\theta}^n\right\}_{n \in \mathbb{N}}$ has the same distribution as $\{\phi \circ \theta^n\}_{n \in \mathbb{N}}$.

Proof. We aim to apply Theorem 1 of [1], whose corrected version is presented in [2]. Let us check that its hypotheses are met in our case. First, the sequence $\left\{L_n^0\right\}_{n \in \mathbb{N}}$ is tight since it converges weakly, which implies with Lemma 5.1 that $\left\{W_n^0\right\}_{n \in \mathbb{N}}$ is tight, since for all $\varepsilon > 0$, there exists M_ε such that for all $n \in \mathbb{N}$,

$$\mathbf{P}^0\left[W_n^0 \leq M_\varepsilon\right] \geq \mathbf{P}^0\left[L_n^0 \leq M_\varepsilon\right] \geq 1 - \varepsilon.$$

Define now on $\Omega \times \mathbb{R}$ the random variables

$$\tilde{W}(\omega, x) := x, \;\; \tilde{\phi}\,(\omega, x) := \phi(\omega),$$

and for all $n \in \mathbb{N}$,

$$\tilde{W}_n(\omega, x) := \tilde{W}\left(\tilde{\theta}^n(\omega, x)\right).$$

Remark that for all $n \in \mathbb{N}$, $\mathfrak{A} \in \mathcal{F}$ and $\mathfrak{B} \in \mathcal{B}(\mathbb{R})$,

$$\mathbf{P}^0 \otimes \delta_0\left[\tilde{\theta}^{-n}\left(\mathfrak{A} \times \mathbb{R}\right)\right] = \mathbf{P}^0 \otimes \delta_0\left[\mathfrak{A} \times \mathbb{R}\right] = \mathbf{P}^0\left[\mathfrak{A}\right]$$

and

$$\begin{aligned}\mathbf{P}^0 \otimes \delta_0\left[\tilde{\theta}^{-n}\left(\Omega \times \mathfrak{B}\right)\right] &= \mathbf{P}^0 \otimes \delta_0\left[\tilde{\theta}^{-n}\left(\tilde{W}_0^{-1}(\mathfrak{B})\right)\right] \\ &= \mathbf{P}^0 \otimes \delta_0\left[\tilde{W}_n^{-1}(\mathfrak{B})\right] = \mathbf{P}^0\left[W_n^0 \in \mathfrak{B}\right].\end{aligned}$$

Hence, the probability distributions $\left\{\mathbf{P}^0 \otimes \delta_0 \circ \tilde{\theta}^{-n}\right\}_{n \in \mathbb{N}}$ on $\Omega \times \mathbb{R}$ have Ω-marginal $\mathbf{P}^0$ and $\mathbb{R}$-marginals the distributions of $\left\{W_n^0\right\}_{n \in \mathbb{N}}$, which form a tight sequence. The sequence $\left\{\mathbf{P}^0 \otimes \delta_0 \circ \tilde{\theta}^{-n}\right\}_{n \in \mathbb{N}}$ is thus tight. On another hand, let us define for all $p \in \mathbb{N}^*$,

(i) $\mathcal{V}_p = \{(\omega, x) \in \Omega \times \mathbb{R};\ D(\omega) < x < D(\omega) + 2^{-p}\}$,

(ii) for any $(\omega, x) \in \Omega \times \mathbb{R}$,

$$f_p(\omega, x) = \mathbf{1}_{x \leq D(\omega)} + \left(-2^p x + 1 + 2^p D(\omega)\right) \mathbf{1}_{(\omega,x) \in \mathcal{V}_p},$$

(iii) for any $(\omega, x) \in \Omega \times \mathbb{R}$,

$$\tilde{\theta}_p(\omega, x) = \left(\theta\omega, [x + f_p(\omega, x)\sigma(\omega) - \xi(\omega)]^+\right).$$

It is then easily checked, that for all p, $\mathcal{V}_p$ is an open set, $\tilde{\theta} = \tilde{\theta}_p$ outside $\mathcal{V}_p$, and that $\tilde{\theta}_p$ is continuous from $\omega \times \mathbb{R}$ into $\mathbb{R}$.

Let us now fix $n, p \geq 1$. We have

$$\frac{1}{n} \sum_{i=0}^{n-1} \left(\mathbf{P}^0 \otimes \delta_0 \circ \tilde{\theta}^{-i}\right)(\mathcal{V}_p) = \frac{1}{n} \sum_{i=-n}^{-1} \mathbf{P}^0 \left[W^0_{-i} \circ \theta^i \in \left(D, D + 2^{-p}\right)\right].$$

But in view of the coupling property of Lemma 2.1, on a $\mathbf{P}^0$-almost sure event $\mathfrak{E}$, there exists $-\infty < \tau(\omega) \leq -1$ such that for all $i \leq \tau$, $L^0_{\tau - i} \circ \theta^i = 0$. In words, given L_0 is null, there exists a.s. a finite instant τ in the past, such that L_τ was null as well. Thus with Lemma 5.1, on $\mathfrak{E}$, $W^0_{\tau-i} \circ \theta^i = 0$ for such an i, so that $W^0_{-i} \circ \theta^i = W^0_{-\tau} \circ \theta^\tau$ since events on $[T_i, T_\tau]$ don't change the value of the workload at time zero whenever the system is empty at T_τ. Hence in view of the previous equality,

$$\begin{aligned} &\frac{1}{n} \sum_{i=0}^{n-1} \left(\mathbf{P}^0 \otimes \delta_0 \circ \tilde{\theta}^{-i}\right)(\mathcal{V}_p) \\ &\qquad \leq \mathbf{P}^0 \left[\left\{\exists j \in [\tau, -1] \cap \mathbb{Z}; W^0_{-j} \circ \theta^j \in \left(D, D + 2^{-p}\right)\right\} \cap \mathfrak{E}\right]. \end{aligned}$$

This implies, letting p tend to infinity in the previous inequality, that

$$\lim_{p \to \infty} \liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \left(\mathbf{P}^0 \otimes \delta_0 \circ \tilde{\theta}^{-i}\right)(\mathcal{V}_p) = 0,$$

which is the last assumption of Theorem 1 of [1] (again, see [2] for the additional assumptions). We can therefore apply this result, yielding that there exists a $\tilde{\theta}$-invariant probability $\tilde{\mathbf{P}}^0$ on $\Omega \times \mathbb{R}$ whose Ω-marginal is $\mathbf{P}^0$. It is now straightforward that

$$\tilde{W}_n(\omega, x) = \phi\left(\theta^{n-1}\omega\right) \circ \phi\left(\theta^{n-2}\omega\right) \circ \ldots \circ \phi(\omega)\{x\},$$

$$\tilde{\phi} \circ \tilde{\theta}^n(\omega, x) = \phi \circ \theta^n(\omega),$$

hence the sequence $\left\{\tilde{W}_n\right\}_{n \in \mathbb{N}}$ satisfies on $\Omega \times \mathbb{R}$ the stochastic recursion

$$\tilde{W}_{n+1} = \tilde{\phi} \circ \tilde{\theta}^n \left\{\tilde{W}_n\right\},$$

where $\left\{\tilde{W}_n, \tilde{\phi} \circ \tilde{\theta}^n\right\}_{n \in \mathbb{N}}$ is stationary under $\tilde{\mathbf{P}}^0$. $\square$

5.4 The loss system G/G/1/1

The stationary workload for the loss system G/G/1/1 queue (there is no place in the buffer, so that each customer is served if and only if he finds an empty system upon arrival), which is constructed in [5], section 2.6, and on an enlarged probability space in [10] and [1] is a particular case of the study presented here, setting

$$D(\omega) = 0, \ \mathbf{P}^0\text{-a.s..}$$

Replacing D by zero in the whole section 5 yields the results mentioned above.

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